Derivation of the Mass-Energy Relation

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The mass equivalent of radiation is implicit in Poincaré's formula for the momentum of radiation, published in 1900, and was used by Poincaré in illustrating the application of his analysis. The equality of the mass equivalent of radiation to the mass lost by a radiating body is derivable from Poincaré's momentum of radiation (1900) and his principle of relativity (1904). The reasoning in Einstein's 1905 derivation, questioned by Planck, is defective. He did not derive the mass-energy relation.

1. INTRODUCTION

THE equation relating mass to energy, $E=mc^2$, appears in two guises. In one guise it applies to radiation existing in space, and is applicable to the interaction of this radiation in pressure and impact phenomena where the radiation retains its identity as such. In these phenomena the "m" in the relation $E=mc^2$ is the *mass equivalent of free radiation.* In the second guise the relation $E=mc^2$ applies to radiation as emitted or absorbed by matter; in this case the "m" is the *mass of matter*, and the significance of the equation is that it describes the gain or loss of mass by matter when absorbing or emitting radiation. If we designate the two masses as m_R and m_M we then have *two* relations

$$E=m_Rc^2$$
$$E=m_Mc^2$$

to be established.

2. POINCARÉ AND THE MASS-ENERGY RELATION

In 1900, in a paper on "The Theory of Lorentz and the Principle of Reaction,"[1] H. Poincaré derived the expression $M=S/c^2$, where M is the *momentum* of radiation, S the *flux* of radiation, and c is the velocity of light. In explaining the significance of this momentum of radiation he said:

"Electromagnetic energy, from the point of view with which we are occupied, behaving like a fluid endowed with inertia, one must conclude that if any apparatus whatever, after having produced electromagnetic energy, sends it by radiation in a certain direction, the apparatus must recoil like a cannon which has launched a projectile ···. It is easy to calculate in figures the amount of this recoil. If the apparatus has a mass of one kilogram, and if it has sent in one direction, with the velocity of light, three millions of joules, the velocity due to the recoil is 1 cm per second."

Consider how Poincaré got this numerical result. He was using his formula, derived in this article, for the momentum of radiation, $M=S/c^2$, and he was putting down the expression for the conservation of momentum in the recoil process. Putting μ for the mass of the recoiling body, and v for its velocity, his working equation is then

$$\mu v=S/c^2.$$

For S, the energy flux, he put the energy E times c, the velocity of light. He then has

$$\mu v=S/c^2=Ec/c^2=E/c^2\cdot c.$$

Inserting his numerical values

$$\mu=10^3 \text{ grams}$$
$$E=3\times10^6 \text{ joules}=3\times10^{13} \text{ ergs}$$
$$c=3\times10^{10} \text{ cm per second,}$$

we get

$$10^3\times v=\frac{3\times10^{13}\times3\times10^{10}}{9\times10^{20}},$$

or

$$v=1.$$

The significant thing for our present study is that Poincaré in this calculation used E/c^2 for the coefficient of c in stating the momentum of radiation, that is, E/c^2 plays the role of mass. The relation $E=m_Rc^2$ was thus contained in his relation $M=S/c^2$.

Let us consider the nature of this "mass" of radiation. It follows from the pressure of radiation as deduced by Maxwell from his electromagnetic theory. Maxwell's formula

$$f=dE/cdt$$

describes the force exerted on an absorbing body by energy received at the rate dE/dt. Now force is also, by definition, the rate of change of momentum of the body, which, by the conservation of momentum, is also the rate of change of momentum of the radiation. We then have that the momentum lost by the radiation is equal to $1/c$ times the energy delivered to the body, or $M_R=E/c$. If now we designate the momentum of radiation by a "mass" m_R times the velocity of the radiation c, we have

$$m_Rc=E/c,$$

or

$$m_R=E/c^2.$$

We thus see that this "mass" of radiation is a concept derived through the definition of force as rate of change of momentum.

[1] H. Poincaré, Arch. néerland. sci. 2, 5, 232 (1900).

Poincaré stated that we may regard electromagnetic energy as a "fluide fictif" of density E/c^2. His momentum of energy is the density of this fictitious fluid times its velocity c. Poincaré discussed and rejected the idea that this fictitious mass was "indestructible," that is, that it was transferred entire in emission or absorption of energy. He decided it must appear as energy in other guises, and said "it is this which prevents us from likening completely this fictitious fluid to a real fluid." In our terminology Poincaré rejected the idea that m_R could become m_M.

In 1904 Poincaré formulated and named the "principle of relativity" according to which it is impossible by observations made on a body to detect its uniform motion of translation.[2] By the use of this principle it is possible to investigate the behavior of the "fluide fictif," and modify the conclusion of Poincaré just quoted.

Consider a body suspended loosely, as by a nonconducting cord, in the interior of an enclosure, the whole system being stationary with respect to the radiation transmitting medium. Let the body emit symmetrically in the "fore" and "aft" directions the amount of energy $\frac{1}{2}E$. The momenta of the two oppositely directed pulses cancel each other, the body does not move, and no information can be obtained as to its change of state.

Now let the whole system of enclosure and suspended particle be set in uniform motion with respect to the radiation transmitting medium with the velocity v. The body now possesses the momentum $mv/[1-(v^2/c^2)]^{\frac{1}{2}}$ and the problem is to determine the effect on this momentum of the two emitted wave trains. Now the energy contents of the two wave trains emitted for the same (measured) period of emission, taking into account the change of frequency of the source and the lengths of the trains,[3] are

$$\frac{E}{2}\frac{[1+(v/c)]}{[1-(v^2/c^2)]^{\frac{1}{2}}} \quad \text{and} \quad \frac{E}{2}\frac{[1-(v/c)]}{[1-(v^2/c^2)]^{\frac{1}{2}}}.$$

The accompanying momenta, from Poincaré's formula, are

$$\frac{E}{2c^2}\frac{[1+(v/c)]}{[1-(v^2/c^2)]^{\frac{1}{2}}}c \quad \text{and} \quad \frac{E}{2c^2}\frac{[1-(v/c)]}{[1-(v^2/c^2)]^{\frac{1}{2}}}c.$$

These being oppositely directed, the net imparted momentum is

$$Ev/c^2[1-(v^2/c^2)]^{\frac{1}{2}}.$$

Forming the equation for the conservation of momentum we have

$$\frac{mv}{[1-(v^2/c^2)]^{\frac{1}{2}}}=\frac{m'v'}{[1-(v'^2/c^2)]^{\frac{1}{2}}}+\frac{Ev}{c^2[1-(v^2/c^2)]^{\frac{1}{2}}},$$

where v' is the velocity of the body after the emission of the radiation.

Now according to Poincaré's principle of relativity, the body must behave in the moving system just as in the stationary system first considered, that is, it does not change its position or velocity with respect to the enclosure, hence $v'=v$, and we get

$$\frac{(m-m')v}{[1-(v^2/c^2)]^{\frac{1}{2}}}=\frac{Ev}{c^2[1-(v^2/c^2)]^{\frac{1}{2}}},$$

giving exactly

$$(m-m')=E/c^2,$$

a relation independent of v, and so holding for the stationary system. The radiating body loses mass E/c^2 when radiating mass E. This is the relation $E=m_Mc^2$.

It thus appears that the mass-energy relation in both its aspects, $E=m_Rc^2$ and $E=m_Mc^2$, is derivable rigorously from the work of Poincaré. His original position, that the mass of the "fluide fictif" must disappear to reappear in other forms of energy, need not have been taken. *Poincaré's inertia of radiation is conserved, and is transformable into or recoverable from mechanical mass.*

The above derivation of the relation $E=m_Mc^2$ was not given by Poincaré himself, and in fact this simple derivation is not met with until long after the relation in question had been established by other, more intricate derivations. The first explicit statement that the heat energy of a body increases its "mechanical" mass was made by F. Hasenöhrl in 1904. Hasenöhrl studied the problem of a hollow enclosure filled with radiation, to determine the effect of the pressure due to the radiation.[4] He showed that "to the mechanical mass of our system must be added an apparent mass $\mu=8E/3c^2$." This he later recalculated as $4E/3c^2$.[5] On the ground that the internal energy of a body must consist in part of radiation Hasenöhrl stated that in general the mass of a body will depend on its temperature.

In 1907 Planck made a more exhaustive study than Hasenöhrl's on the energy "confined" in a body, utilizing Poincaré's momentum of radiation.[6] He found that "through every absorption or emission of heat the inertial mass of a body alters, and the increment of mass

[2] H. Poincaré, "L'etat actuel et l'avenir de la physique mathematique," Congress of Arts and Sciences, St. Louis, Sept. 24, 1904; first published in full in La Revue des Idées, 80, Nov. 15, 1904. For appreciations of the pioneer contributions of Poincaré to the principle of relativity, the formulation of the Lorentz transformations, and the momentum of radiation, see articles by W. Wien and H. A. Lorentz in Acta Math. **38** (1921). See also, Ives, "Revisions of the Lorentz transformations," Proc. Am. Phil. Soc. **95**, 125 (1951).

[3] H. E. Ives, J. Opt. Soc. Am. **34**, 225 (1944).

[4] F. Hasenöhrl, Wien. Sitzungen **IIa**, 113 1039 (1904).

[5] F. Hasenöhrl, Ann. Physik, **4**, 16, 589 (1905). Hasenöhrl noted that this increase of mass was identical with that found twenty years earlier by J. J. Thomson for the case of a charged spherical conductor in motion. For the transformation of the factor $\frac{4}{3}$ to unity upon considering the effect of the enclosure (shell), see Cunningham, *The Principle of Relativity* (Cambridge at the University Press, London, 1914), p. 189.

[6] M. Planck, Sitz. der preuss. Akad. Wiss., Physik. Math. Klasse. **13** (June, 1907).

TABLE I. Mass and energy relations.

		Before emission of radiation Subscript 0	After emission of radiation Subscript 1
Observed from x, y, z platform	Energy of body	E_0	$E_1=E_0-L$
	Radiated energy	0	L
	Mass of body	m	m'
	Momentum of body	0	0
	Kinetic energy of body	0	0
Observed from ξ, η, ζ platform	Energy of body	H_0	$H_1=H_0-L\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$
	Radiated energy	0	$L\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$
	Mass of body	$m\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$	$m'\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$
	Momentum of body	$mv\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$	$m'v\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$
	Kinetic energy of body	$K_0=mc^2\left[1\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}-1\right]$	$K_1=m'c^2\left[1\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}-1\right]$

is always equal to the quantity of heat · · · divided by the square of the velocity of light in vacuo." This derivation of the relation $E=m_Mc^2$ is historically the first valid and authentic derivation of the relation.

Recurring now to the simple derivation above, which depends only on Poincaré's momentum of radiation and his principle of relativity, this appears in an encyclopedic article by W. Pauli in 1920.[7] Since Pauli gives no reference for this treatment, in his otherwise very fully referenced article, it may be presumed to be original with him.[8] In 1933, Becker, in his revision of Abraham's text-book,[9] reproduces Pauli's treatment, with the following comment: "This example is especially of interest, because Einstein with its help derived for the first time the principle of the inertia of energy as a universal law." This comment is incorrect; Einstein, in the work referred to (1905), did not give this derivation; he did not use the momentum of radiation, which is an essential element of this "example," and his derivation was actually incompetent to give the result he announced. This is brought out in the Appendix to the present paper.

3. SUMMARY

Attention is called to the dual aspect of the relation $E=mc^2$, depending on whether the "m" refers to the mass equivalent of free radiation, or the mechanical mass gained or lost through the process of radiation. The first m, designated m_R, was disclosed by Poincaré in his presentation of the momentum of radiation. The second m, designated by m_M, can be obtained from Poincaré's momentum of radiation and his principle of relativity. Historically the first derivation of the relation $E=m_Mc^2$ is to be ascribed to Hasenöhrl and Planck.

APPENDIX. THE 1905 DERIVATION BY EINSTEIN

In 1905 Einstein published a paper with the interrogatory title "Does the Inertia of a Body Depend upon its Energy Content",[10] a question already answered in the affirmative by Hasenöhrl. This paper, which has been widely cited as being the first proof of the "inertia of energy as such," describes an emission process by two sets of observations, in different units, the resulting equations being then subtracted from each other. It should be obvious *a priori* that the only proper result of such a procedure is to give $0=0$, that is, no information about the process can be so obtained. However the fallacy of Einstein's argument not having been heretofore explicitly pointed out, the following analysis is presented:

Einstein sought to derive the mass-energy relation by observing the loss of energy of a radiating body by two sets of observations, one made from a platform stationary with respect to the body, (x, y, z, system), the other from a platform moving with a uniform velocity v with respect to the body, (ξ, η, ζ, system). We shall use the symbols of Einstein's article, adding to them at the start, the accepted formulas for the kinetic energies. The latter are the crux of the problem. The symbols and their relations are most perspicuously set forth in Table I. The problem is to determine, from the data set forth in Table I, the relation between radiated energy (L) and the mass of

[7] W. Pauli, Jr., "Relativitätstheorie," Encyclopedia Math. Wiss. V-2, hft 4, 19, 679 (1920). Pauli assigns momentum to the body not by moving it but by observing it from a moving platform; however, the mathematical formulation is the same as in the treatment here given.

[8] J. Larmor, previously, in considering the case of a radiating body moving through space against the reaction of its own radiation, decided that it would continue at uniform velocity, by losing momentum, at the expense of mass E/c^2. ("On the dynamics of radiation," Proc. Intern. Congr. Math., Cambridge (1912), p. 213; and *Collected Works* (Cambridge University Press, London, England, 1920).

[9] R. Becker, *Theorie der Electronen* (B. G. Teubner, Leipzig, 1930-1933), p. 348.

[10] A. Einstein, Ann. Physik 18, 639 (1905).

the body before and after the emission of radiation (m and m').

We first review Einstein's derivation, accepting his values for the radiation as observed from the x, y, z, and ξ, η, ζ, platforms as L and $L/[1-(v^2/c^2)]^{\frac{1}{2}}$ (these values have been incorporated in the table), and forming by subtraction the equation

$$(H_0-E_0)-(H_1-E_1)=L\left[1\bigg/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}-1\right].$$

After obtaining this equation Einstein introduces the kinetic energies with the statement "$\cdots$ it is clear that the differences $H-E$ can differ from the kinetic energy K of the body with respect to the other system only by an additional constant $C\cdots$ thus we may place

$$H_0-E_0=K_0+C$$
$$H_1-E_1=K_1+C,\text{"}$$

then

$$(H_0-E_0)-(H_1-E_1)=K_0-K_1,$$

and it follows that

$$(K_0-K_1)=L\left[1\bigg/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}-1\right].$$

Neglecting magnitudes of the fourth and higher orders he then gets

$$K_0-K_1=\tfrac{1}{2}Lv^2/c^2.$$

This is the final equation of Einstein's paper. His conclusion as to its physical meaning, namely "if a body gives off the energy L in the form of radiation its mass diminishes by L/c^2" follows from an unstated step, namely

$$K_0-K_1=\tfrac{1}{2}(m-m')v^2,$$

so that

$$\tfrac{1}{2}(m-m')v^2=\tfrac{1}{2}Lv^2/c^2,$$

or

$$m-m'=L/c^2.$$

This is the relation $E=m_Mc^2$, which does not appear explicitly in the paper.

Now it is by no means "clear that, etc." Thus we find Planck in 1907, after deriving the relation in question, as already described, making the following comment:[11] "Einstein has already drawn essentially the same conclusion [Ann. Physik **18**, 639 (1905)] by the application of the relativity principle to a special radiation process, however under the assumption *permissible only as a first approximation*,[12] that the total energy of a body is composed additively of its kinetic energy and its energy referred to a system with which it is at rest."

What Planck objected to was the relation

$$H-E=K+C,$$

or, as he states it

$$H=K+E+C,$$

where, as shown by reference to Table I, H and K are observed from one platform, E from another.

Let us look at this objection (which Planck did not follow up by explanatory analysis). We shall find that what Planck characterized as an assumption permissible only to a first approximation invalidates Einstein's derivation.

Take the relation above derived

$$(H_0-E_0)-(H_1-E_1)=L\left\{\frac{1}{[1-(v^2/c^2)]^{\frac{1}{2}}}-1\right\}.$$

From Table I we have

$$K_0=mc^2\left\{\frac{1}{[1-(v^2/c^2)]^{\frac{1}{2}}}-1\right\}$$

$$K_1=m'c^2\left\{\frac{1}{[1-(v^2/c^2)]^{\frac{1}{2}}}-1\right\},$$

so that

$$K_0-K_1=(m-m')c^2\left\{\frac{1}{[1-(v^2/c^2)]^{\frac{1}{2}}}-1\right\}.$$

By division,

$$(H_0-E_0)-(H_1-E_1)=\frac{L}{(m-m')c^2}(K_0-K_1),$$

which may be considered as the difference of the two relations

$$(H_0-E_0)=\frac{L}{(m-m')c^2}(K_0+C),$$

$$(H_1-{}_1E)=\frac{L}{(m-m')c^2}(K_1+C).$$

Now these are *not*

$$H_0-E_0=K_0+C$$
$$H_1-E_1=K_1+C.$$

They differ by the multiplying factor

$$L/(m-m')c^2.$$

What Einstein did by setting down these equations (as "clear") was to *introduce* the relation

$$L/(m-m')c^2=1.$$

Now this is the very relation the derivation was supposed to yield. It emerges from Einstein's manipulation of observations by two observers because it has been slipped in by the assumption which Planck questioned. The relation $E=m_Mc^2$ was not derived by Einstein.

[11] Reference 6, footnote on p. 566.
[12] My italics, H. E. I.