Apparent Lengths and Times in Systems Experiencing the Fitzgerald-Larmor-Lorentz Contractions

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IN a previous paper[1] I have discussed the phenomena of light signals on moving bodies, when the moving bodies are subject to a contraction of length in the ratio $(1-v^2/c^2)^{\frac{1}{2}}:1$, and when the clocks used are changed in frequency in the same ratio; v being the velocity of the body, and c the velocity of light, both referred to the luminiferous ether. In the present paper I consider, on the same assumptions,—which derive from the phenomena of the aberration of light, and the Michelson-Morley experiment—measurements of lengths and times on relatively moving bodies, in which light signals are not involved. The formulae to be used are in large part already available from the preceding paper, but are here considered from a different aspect.

We shall use in this study an *observation platform*, an *observed body* and an *observed clock*, all in relative uniform motion with respect to each other. The observation platform is distinguished from the observed body only by the fact that we have selected on it two separated observation points, a and b, from which we observe the other body and the clock. The distance apart of these points we determine with a measuring rod, moved past them at some definite observed velocity; the times at these points we establish by similar clocks which have been synchronized and set together when in coincidence and have been moved apart at some definite observed velocity. Our general problem is to find the relationship between length and time intervals on the observation platform and on the observed body and clock, as functions of their relative velocity and the velocities of the rods and clocks used to locate the points a and b on the observation platform. Our general procedure is to use the method of coincidences,—since we are now excluding light signals. We shall make "simultaneous" observations of the points opposite a and b on the observed body, and we shall observe the times of transit of the observed clock past a and b, both by the clocks at a and b, and by the observed clock itself.

The scheme of labeling the various bodies concerned in the precise way just indicated, is adopted in order to emphasize the real distinction between the bodies, which is obscured by the frequent use in discussions of this problem of the term "stationary" to describe the observation platform, and the term "moving" to describe the observed body and clock. The significant distinction is not in terms of rest or motion, since all the bodies are in relative motion with respect to each other, but lies in the nature and point of origin of the *operations* which are performed.

We shall consider first the times of transit of the observed clock between a and b. Let us designate the distance ab by $L_s(1-v^2/c^2)^{\frac{1}{2}}$, where, as in the preceding paper, L_s is the distance ab when the observation platform is stationary in the ether and v is the velocity of the platform through the ether. Let V be the velocity of the clock with respect to the observing platform, its total velocity with respect to the ether being $v+V$. We then have, by Eq. (1) of the previous paper, that the true time of transit is

$$T=\frac{L_s(1-v^2/c^2)^{\frac{1}{2}}}{V}. \tag{1}$$

The time by the observed clock's own indication is

$$T_0=\frac{L_s(1-v^2/c^2)^{\frac{1}{2}}}{V}\left(1-\frac{(v+V)^2}{c^2}\right)^{\frac{1}{2}}. \tag{2}$$

This we must now put in terms of V_0, the velocity of the clock as observed, as discussed in the preceding paper.

Following the procedure there given, we have

$$V=\frac{V_0(1-v^2/c^2)}{(V_0v/c^2)+(1+V_0^2/c^2)^{\frac{1}{2}}}. \tag{3}$$

[1] "Light Signals on Moving Bodies as Measured by Transported Rods and Clocks," J. O. S. A. 27, 263 (1937).

In order to solve (2) we then put $v/c=x$ and $V/c=z$, giving

$$T_0=\frac{(L_s/c)(1-x^2)^{\frac{1}{2}}}{z}\left(1-\frac{(x+z)^2}{c^2}\right)^{\frac{1}{2}}. \tag{4}$$

Putting $V_0/c=\alpha$, we have (from appendix to previous paper)

$$z=\frac{\alpha(1-x^2)}{x\alpha+(1+x^2)^{\frac{1}{2}}} \tag{5}$$

and

$$\left(1-\frac{(x+z)^2}{c^2}\right)^{\frac{1}{2}}=\frac{(1-x^2)^{\frac{1}{2}}}{x\alpha+(1+\alpha^2)^{\frac{1}{2}}} \tag{6}$$

leading to

$$T_0=\frac{(L_s/c)(1-x^2)^{\frac{1}{2}}}{\alpha(1-x^2)/(x\alpha+(1+\alpha^2)^{\frac{1}{2}})}\frac{(1-x^2)^{\frac{1}{2}}}{x\alpha+(1+x^2)^{\frac{1}{2}}}\cdot=\frac{L_s}{cx}=\frac{L_s}{V_0} \tag{7}$$

for the time indicated by the observed clock for its transit between a and b.

We now investigate the transit time as recorded by the two clocks at a and b on the observing platform. It will be the product of the true time $(L_s/V)(1-v^2/c^2)^{\frac{1}{2}}$, multiplied by the rate of either clock $(1-v^2/c^2)^{\frac{1}{2}}$, minus the difference of setting of the clocks due to their separation. If the velocity with which the clock at b was moved from a was W, the difference of setting is

$$\Delta S=\frac{L_s(1-v^2/c^2)^{\frac{1}{2}}}{W}\times\left[\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}-\left(1-\frac{(v+W)^2}{c^2}\right)^{\frac{1}{2}}\right]. \tag{8}$$

We must now express this quantity and the product just specified in terms of the observed velocities V_0 and W_0. Using the results of the two preceding papers[1, 2] we get finally for the time as recorded by the clocks on the observing platform:

$$T_p=\frac{L_s}{V_0}\left(\frac{V_0v}{c^2}+\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}\right)-\frac{L_s}{c}\left(\frac{v}{c}-\frac{1-(1+W_0^2/c^2)^{\frac{1}{2}}}{W_0/c}\right)$$
$$=\frac{L_s}{V_0}\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}+\frac{L_s}{W_0}\left(1-\left(1+\frac{W_0^2}{c^2}\right)^{\frac{1}{2}}\right). \tag{9}$$

Let us now compare (9) and (7). We see at once that the transit time is recorded differently by the observed clock and the clocks on the observing platform. Taking certain typical cases we note that:

(I) If W_0 approximates to 0, the transit time is measured *greater* on the observation platform in the ratio

$$\frac{T_p}{T_0}=\frac{(L_s/V_0)(1+V_0^2/c^2)^{\frac{1}{2}}}{L_s/V_0}=\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}. \tag{10}$$

This may be stated alternatively that the observed clock appears to run slow as evaluated from the observation platform.

(II) If $W_0=V_0$,

$$T_p=T_0, \tag{11}$$

the transit time is the *same* by either observation.

(III) If W_0 is greater than V_0, the transit time is measured *less* on the observation platform.

By inspection of the equations it is evident that we would have obtained identical answers had we interchanged the observing platform and the body to which the observed clock was attached, that is, had we provided another clock on the observed body, and observed one of the original clocks in its transit between the separated clocks of the new observing platform. This means that we have, in the observations (I), (II), and (III) above, certain consequences of the performance of a sequence of *operations*. The observations do *not* tell us that the "observed" clock is fast or slow with respect to a similar clock on the observation platform. They tell us how the "observed" clock indicates with respect to the difference of indication of clocks transported at various velocities between two points on what we have chosen and labeled as the observation platform.

[2] "The Aberration of Clocks and the Clock Paradox," J. O. S. A. **27**, 305 (1937).

We shall next investigate the question of lengths on the observing platform and on the observed body. To do this we imagine marks or traces made from a and b on the passing observed body, these marks being made at the same clock readings at a and b, that is apparently simultaneously. How far apart will these traces measure on the observed body?

The complete solution of this problem is given only by very complicated formulae, for the reason that, in accordance with Eq. (13) of the preceding paper, the divisions which can be laid off on the observation platform and the observed body are functions of the velocities with which the standard measuring rod and the clocks used in the process are moved past the bodies. These could be different for each body, resulting in apparent contractions or expansions over and above those introduced by the relative motion of the observation platform and observed body, which are our chief concern. We shall here merely note this possibility of a variety of obtainable scale values on the two bodies, and confine our attention to the case that on both our bodies divisions have been marked off by placing a standard measuring rod alongside, that is a rod moving at infinitesimal velocity with respect to the body to be marked. With this simplification, Eqs. (7) and (13) of the previous paper give the solution. In terms of our present symbols we have

$$L_0=L_s\left[\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}+\frac{V_0}{W_0}\left(1-\left(1+\frac{W_0^2}{c^2}\right)^{\frac{1}{2}}\right)\right]. \quad (12)$$

As before we consider special cases:

(I)′ If W_0 approximates to 0, the distance measured on the observed body approximates to $L_s(1+V_0^2/c^2)^{\frac{1}{2}}$; that is the distance is measured *greater* than on the observing platform. An alternative statement is that a measuring rod on the observed body acts as though shorter.

(II)′ If $W_0=V_0$, we have

$$L_0=L_s\left[\left(1+\frac{V_0^2}{c^2}\right)+\frac{V_0}{V_0}-\frac{V_0}{V_0}\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}\right]=L_s, \quad (13)$$

the distance between traces measures the *same* as the distance between observing stations.

(III)′ If W_0 is greater than V_0, the traces on the observed body are evaluated as separated *less* than the stations on the observation platform.

We therefore find an exactly parallel condition of affairs as for the comparison of indicated times, and the comments made on that case also apply here. The operations we have gone through, starting with the labeling of one body as the observation platform, picking two observing stations on it, and establishing similar clocks at these stations by transport, lead, by use of the method of coincidences, to traced lengths on the observed body which range from apparent contractions to apparent expansions, depending on the observed velocities of transport of the clocks on the observing platform.

From the standpoint of this series of papers all the results (I), (II), (III), (I)′, (II)′, (III)′ are of coordinate validity and significance. They are observations following upon certain operations performed on bodies subject to the Fitzgerald-Larmor-Lorentz contractions in their motion through the ether. Of these possible observations of lengths and indicated times, those designated I and I', corresponding to infinitesimal velocities of the clocks and rods used to establish distances and times at separated stations on the observation platform, are of interest for the reason that they exhibit, upon a certain choice of descriptive terms, a striking *mimicry* of the length and frequency contractions with respect to the ether, which are the underlying causal phenomena. Thus if we use the term "stationary" to describe the observation platform, and describe the observed bodies and clocks as "in motion with respect to" the observation platform (with the reservation that these terms have no real justification, since all that we can observe is relative motion), we find on going through the operations above described that the "moving" bodies and clocks are observed as contracted in length and frequency in the ratio:

$$\frac{1}{(1+V_0^2/c^2)^{\frac{1}{2}}}:1=\left(1-\frac{V_0^2}{c^2}+\frac{V_0^4}{c^4}-\cdots\right)^{\frac{1}{2}}:1.$$

This mimicry, which holds to the second order of V_0/c, becomes exact if in place of V_0 we express the result in terms of v_{2-1} of a preceding paper,[1] for the condition that the clock on the observing platform has been moved infinitely slowly, for which the symbol v_0, may be used.[3] To do this we put $v_2 - v$, for V; we then have

$$V_0 = \frac{v_2 - v_1}{(1 - v_1^2/c^2)^{\frac{1}{2}}(1 - v_2^2/c^2)^{\frac{1}{2}}} = \frac{(v_2 - v_1)/(1 - (v_1 v_2/c^2))}{(1 - (v_2 - v_1)^2/c^2(1 - (v_1 v_2/c^2)^2)^{\frac{1}{2}}} = \frac{v_0}{(1 - v_0^2/c^2)^{\frac{1}{2}}}. \quad (14)$$

Putting this in the expression for L_0 we have

$$L_0 = L_s\left(1 + \frac{V_0^2}{c^2}\right)^{\frac{1}{2}} = L_s\left(1 + \frac{v_0^2}{c^2(1 - v_0^2/c^2)}\right)^{\frac{1}{2}} = \frac{L_s}{(1 - v_0^2/c^2)^{\frac{1}{2}}}, \quad (15)$$

indicating that the measuring rod carried on the observed body appears shortened in the ratio $(1 - v_0^2/c^2)^{\frac{1}{2}} : 1$. Similarly we find the time interval by the clock on the observed body appears shorter in the same ratio, and hence that the frequency of the observed clock is reduced in that ratio. These apparent contractions thus display a *formal* similarity to those contractions of length and frequency in the ratio $(1 - v^2/c^2)^{\frac{1}{2}} : 1$ due to motion through the ether from which all the results (I), (I)′, (II), (II)′, (III), (III)′, are derived. Physically this means that any results deduced for motion through the ether will be expressible in formally similar equations in terms of relative motion of matter, *provided* that times and lengths be measured by the sequence of operations leading to I and I', and relative velocities by the convention symbolized by v_0.

The Special Theory of Relativity is, in the light of this analysis, a structure built on *the special case* covered by I and I', and the use of the terms "stationary" and "moving" to describe the observation platform and the observed bodies. By virtue of its actual, though unacknowledged, derivation from the ether and the contractions of length and frequency which occur on motion with respect to the ether, it predicts correctly the results of performing *certain selected operations*. Neglecting to inquire into what physical behaviors underly the validity of this special case, *it offers no explanation of what lies behind the particular choice of operations.*[4] It thus gives only a partial account of the phenomena of moving bodies, in contrast to the complete account[5] given by the hypotheses of Fitzgerald, Larmor and Lorentz, based on the phenomena of aberration and the Michelson-Morley experiment. It has an unquestionable *practical* interest, since the terrestrially attainable velocities of rods and clocks are usually negligible compared with the velocity of light, and it is instinctively easy to regard the earth as "stationary." It has the further element of practicality at the present time that, with the current vogue of radio time signals, its times at distant points coincide with these signals under the simple *convention* that these are always to be assigned the constant value of velocity c. A special case of a general theory should not, however, be allowed to usurp the position properly belonging to that general theory.

[3] V_0 and v_0 are two different measures of the relative velocity of the observation platform and the observed bodies. V_0 is obtained from a clock fixed in position on either platform or body, in conjunction with observation of scale divisions going past; v_0 is obtained from the passage of a fixed point on the observed body past two clocks, one transported infinitely slowly to its station.

[4] The constancy of the measured value of the velocity of light—a feature of the conditions I and I'—is introduced by Einstein in his original communication as "in agreement with experience." This "experience" consists of measurements made with clocks and rods moved at infinitesimal speeds with respect to the bodies under observation. Had "experience" been sufficiently wide, i.e., had it included experiments with rapidly moved clocks and rods, it would (if the contractions are in fact of the character here assumed) have demanded qualification or abandonment of this assumption. The restriction to infinitesimal velocities should be openly included in the statement of the second postulate of the Special Relativity Theory, and when so stated the postulate *loses its appearance of fundamental simplicity* and demands a justification. This lack of any explanation is noticed by Levi-Civita (Nuovo Cimento, **13**, 45–65 (1936)) who asks "if it is possible to give any concrete interpretation . . . to the elementary chronotopic interval." The ether and the length and frequency contractions do give a concrete interpretation.

[5] An outstanding merit of the use of the ether as the reference frame is that it handles without difficulty the positive effects obtained in observations with rotating bodies, such as the Sagnac experiment.